The Exponential, Euler's Formula, and the Logarithm

exp, log, and Branches

Lecture Notes

October 1, 2026

1 The Complex Exponential

Definition (complex exponential)

$$\exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

For any $R > 0$, $\dfrac{R^n}{n!} \to 0$ as $n \to \infty$ (an easy exercise). Thus $\exp(z)$ exists for every $z \in \mathbb{C}$, and exp is entire. Differentiating term by term,

$$(e^z)' = \sum_{n=0}^{\infty} \left(\frac{z^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z.$$

The addition law. Fix $a \in \mathbb{C}$ and consider $f(z) := e^z e^{a-z}$. Then

$$f'(z) = e^z e^{a-z} + e^z \cdot (-e^{a-z}) = 0,$$

so $f \equiv f(0) = e^a$: for any $a, z \in \mathbb{C}$, $e^z e^{a-z} = e^a$. Setting $w := a - z$:

Addition law

$$e^z \cdot e^w = e^{z+w} \qquad \text{for all } z, w \in \mathbb{C}.$$

2 Euler's Formula

If $z = x + iy$, then $e^z = e^x e^{iy}$.

Observe. $\overline{e^z} = \sum \overline{z}^n/n! = e^{\bar{z}}$, so $|e^z|^2 = e^z\,\overline{e^z} = e^{z+\bar{z}} = e^{2x}$; hence $|e^z| = e^x$ and, in particular, $|e^{iy}| = 1$: e^{iy} lies on the unit circle.

Euler's Formula

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} = \cos y + i \sin y =: \operatorname{cis}(y).$$

So $y \in \arg e^z$.

In particular, $e^{\pi i} = -1$.

Since $e^{2\pi i} = 1$, we get $e^z = e^{z+2k\pi i}$ for all $k \in \mathbb{Z}$: **the exponential is $2\pi i$-periodic.**

3 The Logarithm

e^w is the complex number with $|e^w| = e^{\operatorname{Re} w}$ and $\operatorname{Im} w \in \arg(e^w)$. To solve $z = e^w$ for w: if $z = 0$, there is no solution ($\log 0$ is undefined). If $z \neq 0$,

$$\operatorname{Re} w = \log|z| \quad \text{(the usual real logarithm of } |z| > 0\text{)}, \qquad \operatorname{Im} w \in \arg z \quad \text{(many possible values!)}.$$

Definition (log, multivalued)

$$\log z := \log|z| + i \arg z$$

is a **multivalued function** of $z \neq 0$.

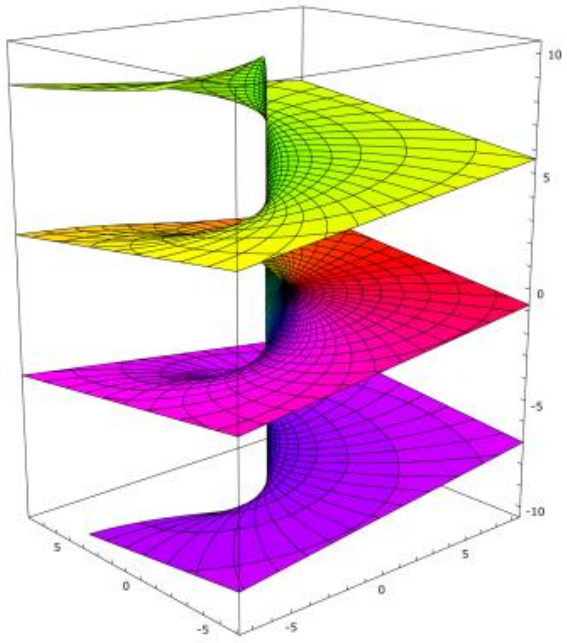


Figure 1: The multivalued logarithm: the graph of $\operatorname{Im} \log z = \arg z$ over the z-plane. Each full turn around 0 climbs one level, by 2π, so the values of $\log z$ fill out an infinite spiral staircase of sheets.

Definition (principal value)

$$\operatorname{Log} z := \log|z| + i \operatorname{Arg} z, \qquad -\pi < \operatorname{Arg} z \leq \pi,$$

is the **principal value** of the logarithm. Log is *not* continuous on $\{\operatorname{Re} z < 0,\ \operatorname{Im} z = 0\}$ (the negative real axis), where $\operatorname{Arg} z$ jumps between values near π and $-\pi$.

Remark. $e^{\operatorname{Log} z} = z$, but $\log(e^z) = \{z + 2\pi i k : k \in \mathbb{Z}\}$.

Definition (branch of the logarithm)

A continuous function $\ell(z)$ on a set K is called a **branch** of log if $\ell(z) \in \log z$ for every $z \in K$.

Example. $\ell(z) = \operatorname{Log} z$ is a branch on the slit plane $\{z : -\pi < \operatorname{Arg} z < \pi\} = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Other branches are obtained by choosing a different 2π-window for the argument – e.g. rotating the branch cut to lie along a different ray produces a different, equally valid, branch on the complementary slit plane.

Interactive illustration. Please have a look at a GeoGebra visualization of the (multivalued) complex logarithm map: `https://www.geogebra.org/m/rVQ4RCYy`.

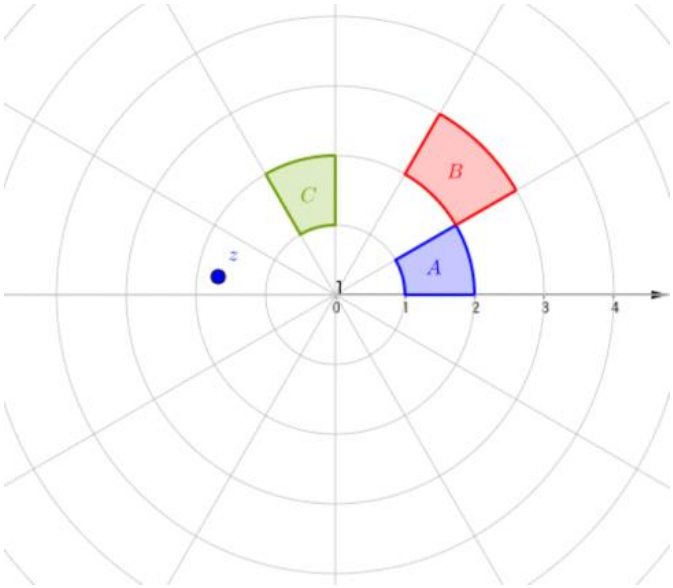


Figure 2: Screenshot of the GeoGebra applet "Complex logarithm map (principal branch)". Live applet: `https://www.geogebra.org/m/rVQ4RCYy`.